HB60

Lie-Poisson dynamics

$\mathfrak{g}$: Lie algebra

Poisson structure on $C^\infty(\mathfrak{g}^*)$: $\{f,g\}_{LP}(\mu) = \langle \mu, [df, dg] \rangle$

$\Longrightarrow$ Lie-Poisson dynamics for Hamiltonian $h \in C^\infty(\mathfrak{g}^*)$:

$$\dot{\mu} = \mathrm{ad}^*_{dh(\mu)}\mu \qquad (*)$$

$\mathfrak{g}^* \simeq \mathfrak{g}$ via biinvariant pairing $\Rightarrow$ $\dot{\mu} = [dh(\mu), \mu]$

Remark: eq. $(*)$ derived for general finite dim. Lie algebras by Poincaré (1901).

Lie-Poisson integrator

Map $\Phi_h : \mathfrak{g}^* \to \mathfrak{g}^*$

(1) $\{f \circ \Phi_h, g \circ \Phi_h\} = \{f, g\} \circ \Phi_h$ (Poisson map)

(2) $\Phi_h(\mu) \in \mathrm{Orb}^*_\mu = \{ \mathrm{Ad}^*_g(\mu) \mid g \in G \}$ (co-ad orbit)

How to construct them?

Lagrangian coordinates viewpoint (cf. Arnold (1966))

Recall Hamiltonian formulation of classical mechanics

Q: configuration manifold

T^*Q : phase space (w. canonical symplectic structure)

$H \in C^\infty(T^*Q)$: Hamiltonian function

$\Rightarrow$ Hamilton's eq. of motion

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{cases} \qquad (**)$$

Connection between $(*)$ and $(**)$?

What happens if:

(1) Q is a Lie group G

(2) Hamiltonian invariant under action of G on T^*G:

$$H(\underbrace{g\cdot(q,p)}_{(gq,(g^{-1})^*p)}) = H(q,p) \quad \forall g\in G,\ (q,p)\in T^*G$$

$\Rightarrow$ Reduction theory: dynamics evolve on T^*G/G How to work with quotient space?

G Lie group $\Rightarrow$ TG trivial (by left or right translation)

$$\Rightarrow T^*G \simeq G\times\mathfrak{g}^* \text{ via } (q,p)\longmapsto (q,\underbrace{q^*p}_{\equiv\mu})$$

momentum map

Action on $G \times \mathfrak{g}^*$: $g \cdot (q, \mu) = (gq, (gq)^*(g^{-1})^* p)$

$= (gq, \mu)$

$\Rightarrow \quad \underline{\underline{T^*G/G}} \simeq G \times \mathfrak{g}^*/G = G/G \times \mathfrak{g}^* \simeq \underline{\underline{\mathfrak{g}^*}}$

Co-tangent bundle reduction theory:

T^*G/G admits Poisson structure inherited from T^*G

Projection $T^*G \to \mathfrak{g}^*$ (momentum map)

Poisson submersion

In plain words: every LP-system on $\mathfrak{g}^*$ corresponds to canonical Hamiltonian system on T^*G w.r.t G-invariant Hamiltonian

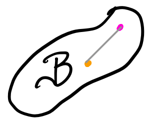
... and vice versa

Interpretation by V. Arnold (1966):

Hamiltonian quadratic and pos. def. $\Rightarrow$ integrated curve in G is geodesic

w.r.t. left (or right) invariant Riemannian metric

Example 1: free rigid body



body $\mathcal{B}$ of infinitesimal mass particles constrained to preserve pairwise lengths $\Rightarrow Q = SO(3)$

Newtonian mechanics $\Rightarrow$ kinetic energy

$$L(R,\dot{R}) = \frac{1}{2}\int_{\mathcal{B}} |\dot{R}x|^2\,dx = \frac{1}{2}\int_{\mathcal{B}} |\underbrace{R^{-1}\dot{R}}_{\hat{\omega}\in\mathfrak{so}(3)}\, x|^2\,dx = \omega\cdot\mathbb{I}\omega$$

moments of inertia

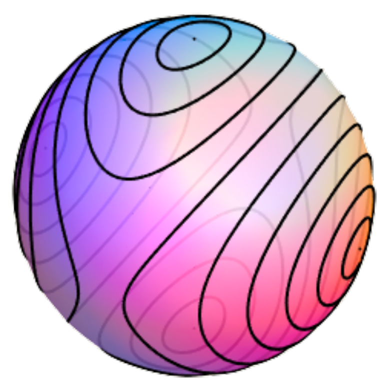
angular velocity vector

$\Rightarrow$ LP-system on $\mathfrak{so}(3)^*$:

$\mu = \mathbb{I}\,\omega$ (Legendre transform)

$$\dot{\mu} = \mathbb{I}^{-1}\mu \times \mu$$

angular momentum vector



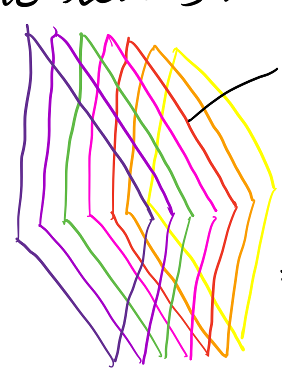
Conservation laws:

(1) total angular momentum $\frac{1}{2}\|\mu\|^2$

(2) energy $\frac{1}{2}\omega\cdot\mu$

Where does conservation of momentum come from?

Weinstein (1983): Poisson manifolds foliated into symplectic leaves:



each leaf symplectic structure

System evolves on leaf determined by initial condition

$\Rightarrow$ functions constant on leafs conserved (Casimir functions)

Example 2: point vortices on S^2

leaves for LP-system: co-adjoint orbits

Co-adjoint orbits of $\mathfrak{so}(3)^* =$ spheres

$\Longrightarrow$ LP on $\mathfrak{so}(3)^* \simeq \mathbb{R}^3$ convenient way to describe Hamiltonian dynamics on S^2 (and $(S^2)^N$)

vortex strengths

$$H: \mathbb{R}^{3N} \to \mathbb{R}, \quad H(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \frac{1}{4\pi} \sum_{i \neq j} \Gamma_i \Gamma_j \log(1 - \mathbf{r}_i \cdot \mathbf{r}_j)$$

Dynamics:

$$\boxed{\dot{\mathbf{r}}_i = \frac{1}{2\pi} \sum_{i \neq j} \Gamma_j \frac{\mathbf{r}_i \times \mathbf{r}_j}{1 - \mathbf{r}_i \cdot \mathbf{r}_j}} \qquad (***)$$

Conservation laws:

$\|\mathbf{r}_i\|^2$ (Casimirs)

H (Energy)

$\mathbf{J} = \sum_{i=1}^{N} \Gamma_i \mathbf{r}_i$ (total angular momentum)

Nöther quantity (not Casimir)

Example 3: geodesics on symplectomorphisms

M closed symplectic manifold $\Rightarrow \mathfrak{g} = C^\infty(M)$ Poisson algebra

L^2-inner product biinvariant: $\langle \{f,g\}, h\rangle_{L^2} = \langle f, \{g,h\}\rangle_{L^2}$

Hamiltonian $H(\omega) = \frac{1}{2}\int_M \omega \Delta^{-1}\omega$

$\Rightarrow$ LP on $\mathfrak{g}^* \simeq C^\infty(M)$: $\dot\omega = \{\psi, \omega\}$, $\Delta\psi = \omega$ $(****)$

Special case: $M = S^2 \Rightarrow$ vorticity formulation of Euler eq. on S^2

Casimirs: $C_f(\omega) = \int_M f(\omega(x))\,dx$ $\quad f: \mathbb{R} \to \mathbb{R}$

Connection to point-vortices:

$\omega = \sum_{i=1}^N \Gamma_i \delta_{r_i}$ weak solution of $(****)$

(Helmholtz, Kirchhoff, Gröbli, Poincaré)

L. Onsager's idea (1949)

(1) "approximate" ω by large number of PV

(2) apply statistical mechanics to understand long-time behaviour

statistical hydrodynamics

Predictions

System settles about steady vortex condensates of equal signed PV

$\boxed{\text{PROBLEMS}}$

(1) Non-regular (weak) $\Rightarrow$ no higher order Casimirs

BUT: conservation of Casimirs affect dynamics

(2) Numerical simulations contradict steady condensates $\Rightarrow$ "quasi-periodic" behaviour

V. Zeitlin's idea (1991)

Use quantization theory to replace (****) by finite dim. Lie-Poisson system

Quantization in a nutshell (i.e. $M = S^2$)

Vorticity state $\omega : S^2 \to \mathbb{R}$

Aim: mapping $T_N : \omega \longmapsto W \in \mathfrak{u}(N)$ such that

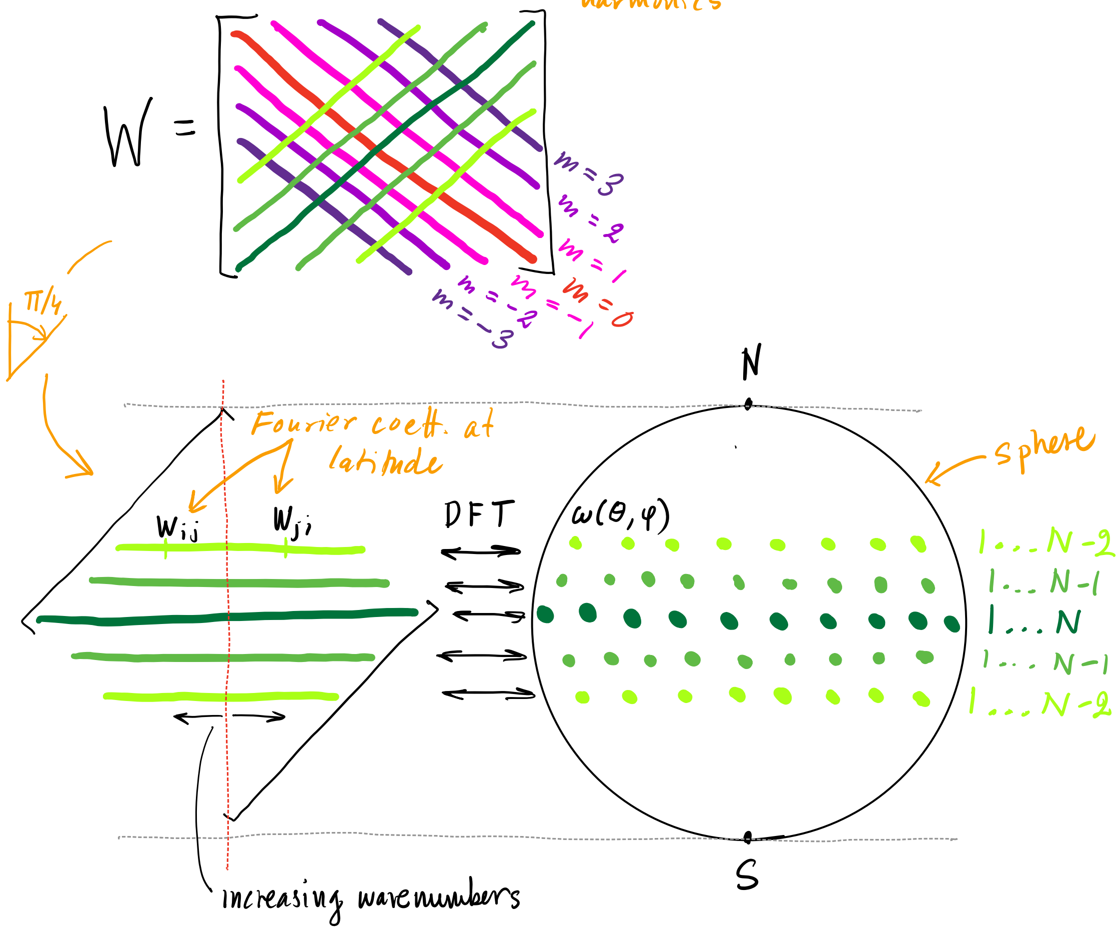
$$T_N(\{\omega_1, \omega_2\}) \approx \frac{1}{\hbar}[T_N\omega_1, T_N\omega_2]$$

matrix commutator

Hoppe (1989): explicit quantization on $\mathbb{T}^2$ and $\mathbb{S}^2$

Route: (for $\mathbb{S}^2$) $\omega \longmapsto \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \omega_{\ell m} Y_{\ell m} \longmapsto \underbrace{\sum_{\ell=0}^{N} \sum_{m=-\ell}^{\ell} \omega_{\ell m} T^N_{\ell m}}_{\in\, \mathfrak{u}(N)}$

spherical harmonics



Euler-Zeitlin equation

$$\omega \in L^\infty(\mathbb{S}^2) \longleftrightarrow W \in \mathfrak{u}(N)$$

$$\Delta \psi = \omega \longleftrightarrow \Delta_N P = W$$

$$\Delta_N P = \sum_{i=1}^{3} [[P, X_i^N], X_i^N]$$

Hoppe-Yau Laplacian

$$\dot{\omega} = \{\psi, \omega\} \longleftrightarrow \dot{W} = [P, W]$$

Euler-Zeitlin equation

stream matrix

vorticity matrix

Back to numerical integration

$\dot{W} = [P, W]$

$W, P \in \mathfrak{su}(N)$

possible approaches

Splitting

Flow evolves on orbits
$\Rightarrow$ Lie group methods / RKMK

symplectic?

Flow isospectral
$\Rightarrow$ Isospectral methods

"Unreduced" flow on $T^*SU(N)$
$\Rightarrow$ Moser–Veselov (constrained dyn.)
$\Rightarrow$ exp-based VI

PROBLEMS:

exp/dexp/inv :

non-symplectic (maybe) :

non-parallell :

Engö & Faltinsen (2001)

The criterion for a good numerical integrator of Lie–Poisson systems becomes the following. The method should produce numerical approximations that stay on the coadjoint orbits, and conserve the Casimirs. Further, the method should *either* conserve the energy *or* preserve the Lie–Poisson structure.

We present methods that automatically retain the two first features. Within this class there exist integrators that also preserve the energy. Conserving the Lie–Poisson structure is a more delicate problem and will not be treated here.

Connection between Lie group methods and isospectral

$\dot{z} = X_H(z)$

$\updownarrow$

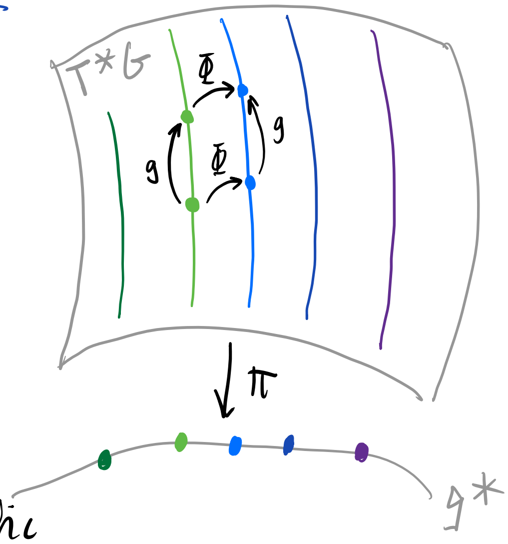
isospectral

flow on Lie group

$$\dot{\mu} = ad^*_{\rho}\mu \quad \xleftrightarrow{\mu = g\cdot\mu_0} \quad \dot{g} = g\cdot\rho$$

Discrete reduction theory

$$\begin{tikzcd}
T^*G \arrow[r, "\Phi_{h\circ\pi}"] \arrow[d, "\pi"] \arrow[dr, phantom, "?"] & T^*G \arrow[d, "\pi"] \\
\mathfrak{g}^* \arrow[r, "\varphi_h"] & \mathfrak{g}^*
\end{tikzcd}$$



Theorem: $\mathfrak{g} \subseteq \mathfrak{gl}(N,\mathbb{C})$ quadratic

$\Longrightarrow$ every SRK gives $\underbrace{\text{LP-integrator}}_{\text{isospectral}}$ on $\mathfrak{g}^*$

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

Formula for $\mathfrak{g} = \mathfrak{su}(N)$: $\dot{W} = [B(W), W]$

$$X_i = -h\Big(W_n + \sum_{j=1}^{s} a_{ij} X_j\Big) B(\tilde{W}_i)$$

$$K_{ij} = h B(\tilde{W}_i)\Big(\sum_{k=1}^{s}\big(a_{ik} X_k + a_{jk} K_{ik}\big)\Big)$$

$$\tilde{W}_i = W_n + \sum_{j=1}^{s} a_{ij}\big(X_j - X_j^* + K_{ij}\big)$$

$$W_{n+1} = W_n + h\sum_{i=1}^{s} b_i [B(\tilde{W}_i), \tilde{W}_i]$$

$i,j = 1,\dots,s$

Example (implicit midpoint)

$$\boxed{\begin{aligned} W_n &= \Big(I - \tfrac{h}{2}B(\tilde{W})\Big)\tilde{W}\Big(I + \tfrac{h}{2}B(\tilde{W})\Big) \\ W_{n+1} &= \Big(I + \tfrac{h}{2}B(\tilde{W})\Big)\tilde{W}\Big(I - \tfrac{h}{2}B(\tilde{W})\Big) \end{aligned}}$$

minimal variables: $W_n, \tilde{W}, W_{n+1} \in \mathfrak{su}(N)$