Brief history of geometric mechanics

Klas Modin, Chalmers & GU

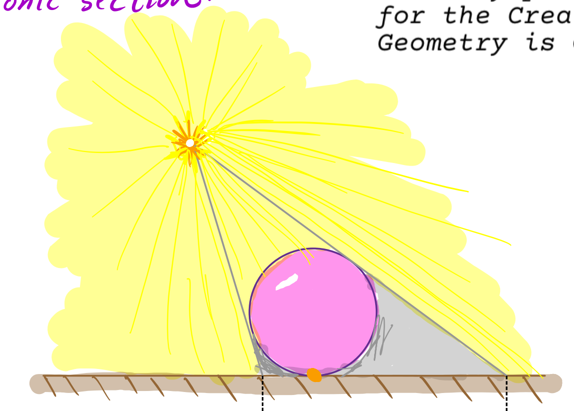
Overview

- Kepler, Newton, Euler (1609 – 1757)
  - Elliptic orbits
  - Newton's force law
  - Motion of incompressible perfect fluid
- Euler, Lagrange, Hamilton (1755 – 1833)
  - Calculus of variations
  - Euler–Lagrange equations
  - Hamilton's principle
- Lie, Poincaré, Nöther, Arnold (1888 – 1966)
  - Phase space, symplectic geometry
  - King Oscar II's prize
  - Theory of transformation groups
  - Nöther's theorem
  - Lie–Poisson dynamics
  - Riemannian interpretation of Euler's eq.

Kepler's laws and Euclidean geometry

"Geometry existed before the Creation. It is co-eternal with the mind of God.
Geometry provided God with a model for the Creation.
Geometry is God himself."

Conic section:



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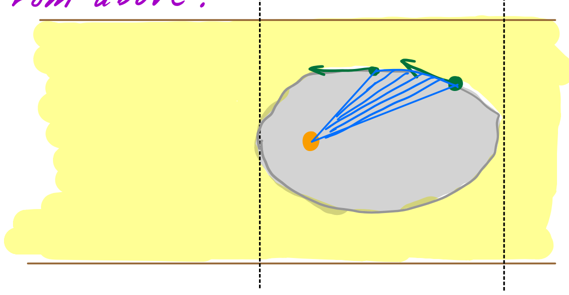


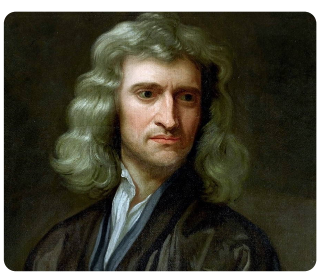
- First law: planets move in elliptical orbits

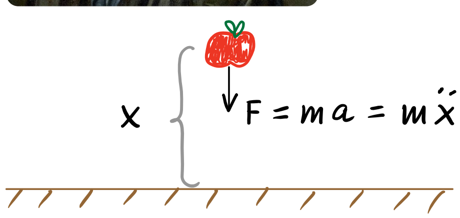
- Second law: equal times sweep equal areas

- Third law: T = orbital period
  R = average distance

$$\frac{T^2}{R^3} = \text{const}$$

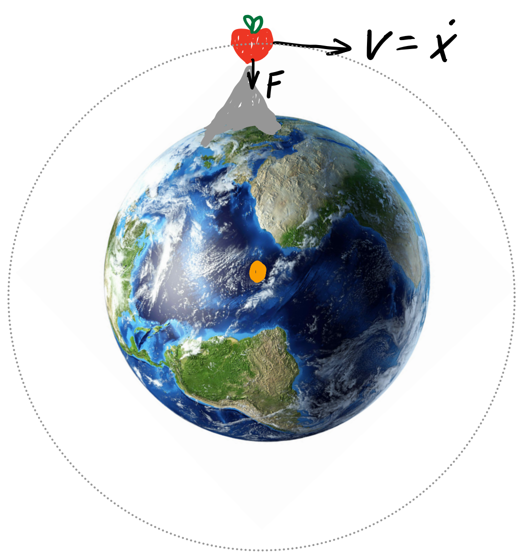
Newton's second law



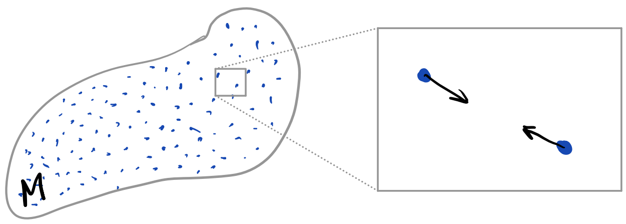


$$x \qquad F = ma = m\ddot{x}$$

$$\boxed{F \sim \frac{m M}{d^2}}$$



Incompressible Euler equations



Fluid motion: Newton's equations for infinitely many fluid particles

"At the same time, it cannot be that the motion of all particles of the fluid is bound in *no* way by any law; nor can *any* conceivable motion of a single particle be allowed. For since the particles are impenetrable, it is clear that no motion can take place where some particles go through others, or that they penetrate each other. An infinite number of such motions should be excluded, and only the remaining are to be considered, and clearly the task is to determine by which property these remaining possibilities can be distinguished by each other."

Euler's approach: (1) characterize "possible motions"
(2) among those, select the true ones from principles of mechanics

...in modern language:

(1) determine constraint manifold

(2) formulate Newton's eq. with constraint forces

Euler's insight: describe vectorfield $v(x,t)$ in which particles travel

$$\Rightarrow \sum_{i=1}^{d} \frac{\partial v^i}{\partial x^i} = 0 , \qquad \boxed{\operatorname{div} v = 0} \qquad (1)$$

$$\Rightarrow \frac{\partial v^i}{\partial t} + \sum_{j=1}^{d} v^j \frac{\partial v^i}{\partial x^j} = -\frac{\partial p}{\partial x^i} , \qquad \boxed{\dot{v} + \nabla_v v = -\nabla p} \qquad (2)$$

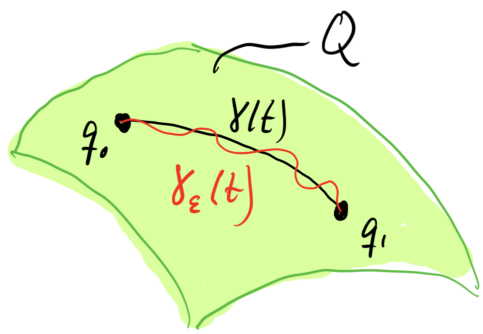
Calculus of variations

Analytical mechanics: reformulation of Newtonian mechanics based on variational principle

Setting:

Q: configuration space (manifold)



$L(q,\dot q)$: Lagrangian function ($TQ \longrightarrow \mathbb{R}$)

Hamilton's principle:

action $\quad S(\gamma) = \displaystyle\int_{t_0}^{t_1} L(\gamma(t),\dot\gamma(t))\,dt$ extremized

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S(\gamma_\varepsilon(t)) = 0 \quad \text{for } \gamma_0(t) = \gamma(t)$$

Euler-Lagrange equations:

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S(\gamma_\varepsilon(t)) = \int_{t_0}^{t_1} \Big[\frac{\partial L}{\partial q} \cdot \underbrace{\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \gamma_\varepsilon(t)}_{\delta q} + \frac{\partial L}{\partial \dot q} \cdot \underbrace{\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \dot\gamma_\varepsilon(t)}_{\frac{d}{dt}\delta q} \Big] dt =$$

$$= \int_{t_0}^{t_1} \Big[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot q} \frac{d}{dt} \delta q \Big] dt = \int_{t_0}^{t_1} \Big[\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot q} \Big] \delta q \, dt + \cancelto{0}{b.t.}$$

$\Rightarrow$ Euler-Lagrange eq:
$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = 0} \quad (*)$$

Special case:

$Q = \mathbb{R}^n$

$$L(q, \dot q) = \frac{m}{2} |\dot q|^2 - V(q) \Rightarrow \boxed{m \ddot q = -\nabla V(q)} \quad (**)$$

Newton's eq. for potential $V(q)$

Note: $(*)$ valid in <u>any</u> coordinates on Q

$(**)$ requires <u>affine</u> coordinates

Hamilton's equations

Conservation law: $E(q,\dot q) = \left\langle \frac{\partial L}{\partial \dot q}, \dot q \right\rangle - L(q,\dot q)$

$$\frac{d}{dt}E = \left\langle \frac{d}{dt}\frac{\partial L}{\partial \dot q}, \dot q \right\rangle + \cancel{\left\langle \frac{\partial L}{\partial \dot q}, \ddot q \right\rangle} - \left\langle \frac{\partial L}{\partial q}, \dot q \right\rangle - \cancel{\left\langle \frac{\partial L}{\partial \dot q}, \ddot q \right\rangle}$$

$$= \cancel{\left\langle \frac{\partial L}{\partial q}, \dot q \right\rangle} - \cancel{\left\langle \frac{\partial L}{\partial q}, \dot q \right\rangle} = 0$$

Hamilton's calculation: assume $\dot q \longmapsto \frac{\partial L}{\partial \dot q}(q,\dot q)$ bijective

Momentum $p := \frac{\partial L}{\partial \dot q} \implies \dot p = \frac{\partial L}{\partial q}$

Hamiltonian $H\underset{\substack{\rotatebox{90}{$\in$}\\ T^*Q}}{(q,p)} = E\Big(q, \underbrace{\frac{\partial L}{\partial \dot q}^{-1}(p)}_{\dot q}\Big) = \langle p, \dot q\rangle - L(q,\dot q)$

$$\frac{\partial H}{\partial q} = \cancel{\left\langle p, \frac{\partial \dot q}{\partial q}\right\rangle} - \frac{\partial L}{\partial q} - \cancel{\left\langle \underbrace{\frac{\partial L}{\partial \dot q}}_{p}, \frac{\partial \dot q}{\partial q}\right\rangle} = -\frac{\partial L}{\partial q}$$

$$\frac{\partial H}{\partial p} = \dot q + \cancel{\left\langle p, \frac{\partial \dot q}{\partial p}\right\rangle} - \cancel{\left\langle \underbrace{\frac{\partial L}{\partial \dot q}}_{p}, \frac{\partial \dot q}{\partial p}\right\rangle} = \dot q$$

$$\implies \boxed{\begin{cases} \dot q = \dfrac{\partial H}{\partial p} \\ \dot p = -\dfrac{\partial H}{\partial q} \end{cases}}$$

Hamilton's equations

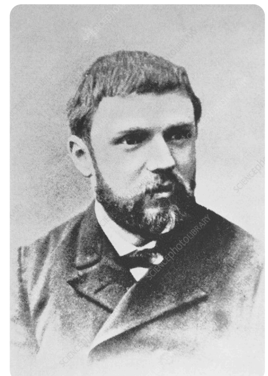
$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad \boxed{\dot z = J^{-1}\nabla H(z)}$

$z = \begin{bmatrix} q \\ p \end{bmatrix}$

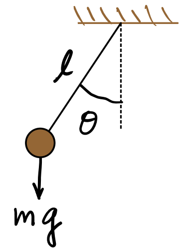
Note: • very symmetric form: q and p dual variables

Question: which change of coordinates preserves this form?

Phase space and symplectic geometry

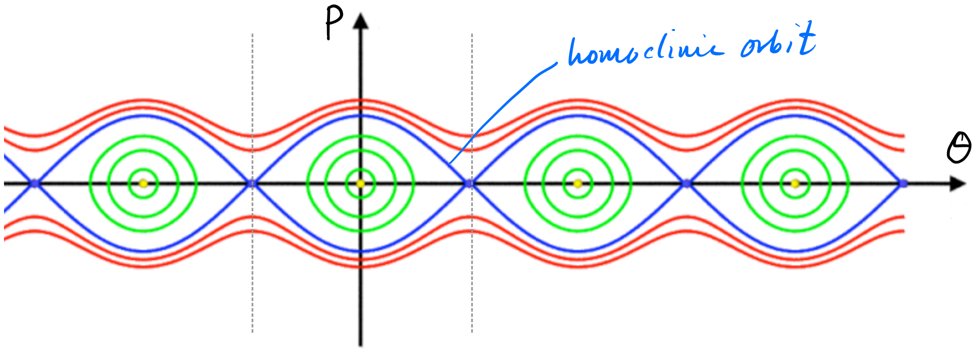


Example: pendulum



$$H(\theta, p) = \frac{1}{2} p^2 - \cos\theta$$

angular momentum
$p = m\ell^2\dot{\theta}$



$Q = S^1$
$T^*Q =$ cylinder

Just by looking: • stable equilibrium at $(0,0)$
• unstable equilibrium at $(\pi, 0)$

Prize competition of King Oscar II

Text by Mikael Rågstedt

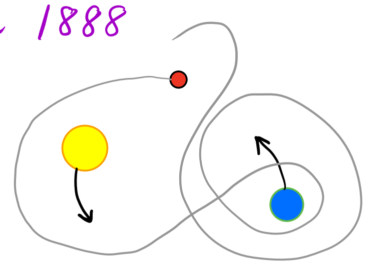
www.mittag-leffler.se/library/henri-poincare

Mittag-Leffler 1884: prize in honour of King Oscar II

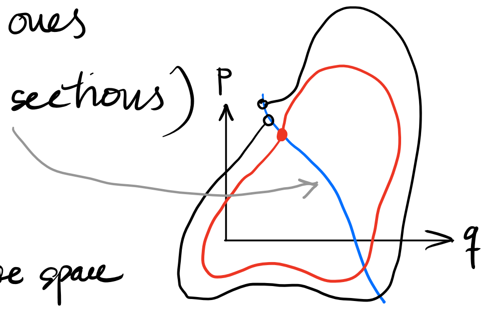
Problem 1

For a system of arbitrarily many mass points that attract each other according to Newton's law, assuming that no two points ever collide, find a series expansion of the coordinates of each point in known functions of time converging uniformly for any period of time.

Poincaré's "solution" from June 1888



- Restricted 3-body problem
- Analysis heavily based on phase portraits
- Trajectories close to periodic ones
  $\Rightarrow$ return maps (Poincaré sections)



- Recurrence theorem
  solutions return arbitrary close in finite time for compact phase space
- Integral invariants

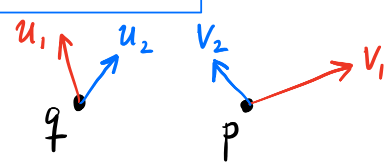
piece of 2-d surface S in phase space T^*Q

$\Phi_t : T^*Q \to T^*Q$ phase flow $\left(t \mapsto \Phi_t(q_0, p_0)\right)$

solution

$$\boxed{\sum_{i=1}^{n} \int_S dq^i dp_i = \sum_{i=1}^{n} \int_{\Phi_t(S)} dq^i dp_i}$$

Infinitesimal version:



$$[u_1 \;\; v_1] \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{J} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = [u_1 \;\; v_1]\, D\Phi_t(q,p)^T \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} D\Phi_t(q,p) \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}$$

$$\Rightarrow \quad \boxed{D\Phi_t^T \, J \, D\Phi_t = J}$$

geometric notation
$\omega = \sum dq^i \wedge dp_i$
$\Phi_t^* \omega = \omega$

flowmap Φ_t is symplectic

Hamiltonian dynamics reformulated

(M, ω) symplectic manifold

closed, non-degenerate 2-form

Hamiltonian vectorfield on M for $H \in C^\infty(M, \mathbb{R})$:

$$\boxed{\omega(X_H, \cdot) = dH}$$

$$F \in C^\infty(M, \mathbb{R}) \text{ first integral} \iff \omega(X_H, X_F) = 0$$

$$\{H, F\} = 0$$

Poisson bracket

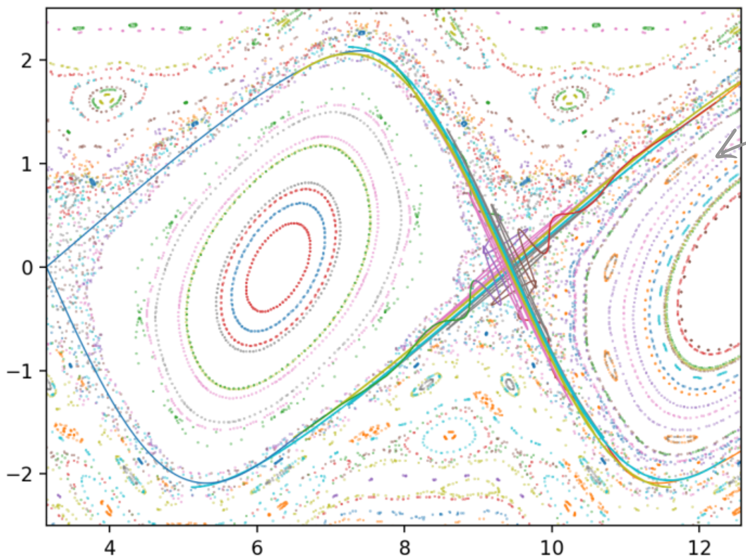
... Poincaré announced winner, clearly ...

... BUT...

...Poincaré found an error!

Long story short: Poincaré resubmitted Jan 1890

- Homoclinic orbits
- Reverse conclusion: instable motion can occur
  $\Rightarrow$ chaotic motion
- Homoclinic tangle (stretch and fold)



Poincaré section

Continous transformation groups

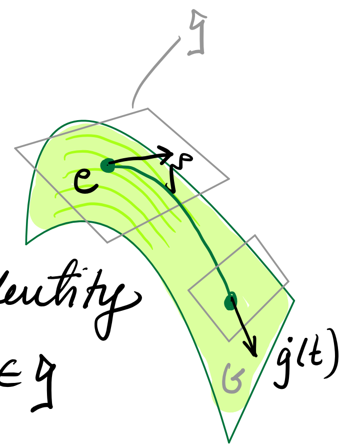
Lie group G:

- manifold
- group: $g, h \in G \Rightarrow gh \in G$
  $e \in G$ $\quad g^{-1} \in G$
- compatibility
  (1) $(g,h) \mapsto gh$ smooth
  (2) $g \mapsto g^{-1}$ smooth

Examples:

- $GL(n)$, all invertible $n\times n$ matrices
- $SL(n) = \{A \in GL(n) \mid \det(A) = 1\}$
- $O(n) = \{A \in GL(n) \mid A^T A = I\}$
- $U(n) = \{A \in GL(n,\mathbb{C}) \mid A^* A = I\}$

Lie algebra $\mathfrak{g}$ of G:



- $\mathfrak{g} = T_e G$, tangent space at identity
- elements in G generated by $\xi \in \mathfrak{g}$

  $\dot{g}(t) = g(t)\,\xi$, $\quad g(0) = e$

  $g(s+t) = g(s)\,g(t)$ (one-parameter subgroup)

- when does two such flows commute?

  $\dot{h}(s) = h(s)\,\eta$, $\quad h(0) = e$

  Condition: $g(t)\,h(s) = h(s)\,g(t)$

  $\iff h(s)^{-1} g(t) h(s)\, g(t)^{-1} = e$

  Differentiate at $s=0$ & $t=0$:

  $\Rightarrow [\xi, \eta] = 0$ ← Lie bracket

  $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$

In basis $e_1, \ldots, e_n$ of $\mathfrak{g}$:

$$[\xi,\eta] = \sum_{i,j,k} c^k_{ij}\, \xi^i \eta^j e_k$$

structure constants $c^k_{ij} = [e_i, e_j]\cdot e_k$

Matrix groups:

$$[\xi,\eta] = \xi\eta - \eta\xi$$

commutator

Symmetry groups and conservation laws

Lie group G acts on Q:

$$(g,q) \longmapsto g\cdot q \in Q$$

$$(hg)\cdot q = h\cdot(g\cdot q)$$

Nöther (1915):

Symmetric Lagrangian $L(q,\dot q) = L(g\cdot q, g\cdot \dot q)$ $\forall g \in G$

$$\Longrightarrow \quad I_\xi(q,\dot q) = \left\langle \frac{\partial L}{\partial \dot q}, \xi\cdot q \right\rangle \text{ conserved } \forall \xi \in \mathfrak{g}$$

Hamiltonian description:

$$H(q,p) = H(g\cdot q, g\cdot p) \quad \forall g \in G$$

$\Rightarrow$ momentum map $I : T^*Q \longrightarrow \mathfrak{g}^*$ conserved

$$\langle I(q,p), \xi \rangle = \langle p, \xi\cdot q \rangle$$

$$0 = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} H(g(\varepsilon)\cdot q, g(\varepsilon)\cdot p) = \left\langle \frac{\partial H}{\partial q}, \xi\cdot q \right\rangle - \left\langle p, \xi\cdot \frac{\partial H}{\partial p} \right\rangle$$

$$\Rightarrow \frac{d}{dt}\langle I(q,p), \xi \rangle = \langle \dot p, \xi\cdot q \rangle + \langle p, \xi\cdot \dot q \rangle$$

$$= \left\langle -\frac{\partial H}{\partial q}, \xi\cdot q \right\rangle + \left\langle p, \xi\cdot \frac{\partial H}{\partial p} \right\rangle = 0$$

Question: what if $Q = G$?

Lie-Poisson dynamics

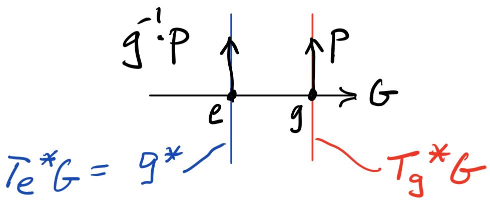
used variational principle, here Hamiltonian description

Poincaré (1901): Hamiltonian system on T^*G

Symmetric Hamiltonian $\quad H(g,p) = H(hg, h\cdot p) \ \forall h \in G$

Dynamics on $T^*G/G = \{(Gg, G\cdot p) \mid (g,p) \in T^*G\}$

How work with cosets $(Gg, G\cdot p)$?



$\Rightarrow T^*G/G \simeq \mathfrak{g}^*$

reduced Hamiltonian

$$\Rightarrow H(g,p) = \bar{H}(\underbrace{g^{-1}\cdot p}_{\mu}) \qquad \bar{H} : \mathfrak{g}^* \to \mathbb{R}$$

$\leftarrow$ new variable

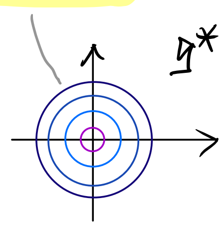
$(g,p) \longleftrightarrow (g,\mu)$ Hamilton's eq. in μ ?

Lie-Poisson system:

$$\boxed{\dot{\mu} = ad^*_{d\bar{H}(\mu)}(\mu)} \qquad ad_\xi = [\xi, \cdot]$$

evolves on co-adjoint orbits

$$\boxed{\dot{\mu}_i = \sum_{jk} C_{ij}^k \xi^j \mu_k, \ \xi = d\bar{H}(\mu)}$$

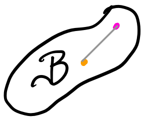


Reconstruction:

$\dot{g}(t) = g(t)\cdot \xi(t)$

$p(t) = g(t)\cdot \mu(t)$

Example: free rigid body



body $\mathcal{B}$ of infinitesimal mass particles constrained to preserve pairwise lengths $\Longrightarrow Q = SO(3)$

Newtonian mechanics $\Longrightarrow$ kinetic energy

$$L(R,\dot R) = \frac{1}{2}\int_{\mathcal{B}} |\dot R x|^2\, dx = \frac{1}{2}\int_{\mathcal{B}} |\underbrace{R^{-1}\dot R}_{\hat\omega \in \mathfrak{so}(3)} x|^2\, dx = \omega \cdot \mathbb{I}\omega$$

moments of inertia

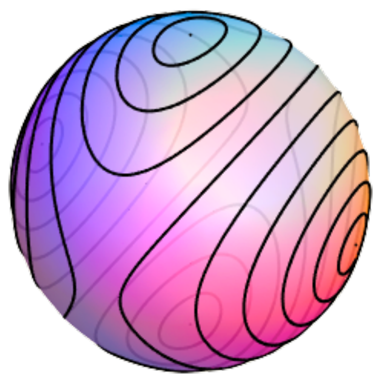
angular velocity vector

$\Longrightarrow$ LP-system on $\mathfrak{so}(3)^*$:

$\mu = \mathbb{I}\,\omega$ (Legendre transform)

$\dot\mu = \mathbb{I}^{-1}\mu \times \mu$ $\quad H(\mu) = \mu \cdot \mathbb{I}^{-1}\mu$

angular momentum vector



Conservation laws:

(1) total angular momentum $\frac{1}{2}\|\mu\|^2$

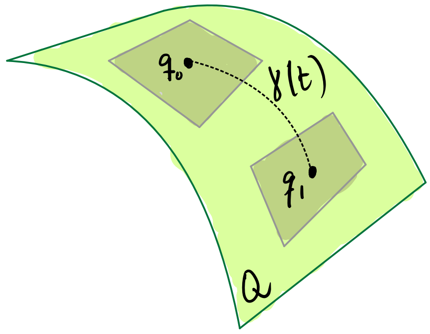
(2) energy $\frac{1}{2}\omega \cdot \mu$

Where does conservation of momentum come from?

it's a Casimir function!

Geodesics on diffeomorphisms and hydrodynamics

Riemannian manifold



assign inner product $\langle \cdot, \cdot \rangle_q$ to T_qQ

Length of γ: $\ell(\gamma) = \int_{t_0}^{t_1} \sqrt{\langle \dot\gamma(t), \dot\gamma(t) \rangle_{\gamma(t)}}\, dt$

γ geodesic $\Leftrightarrow$ $\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \ell(\gamma_\varepsilon) = 0$

Kinetic energy systems

Lagrangian $L(q, \dot q) = \frac{1}{2} \langle \dot q, \dot q \rangle_q$

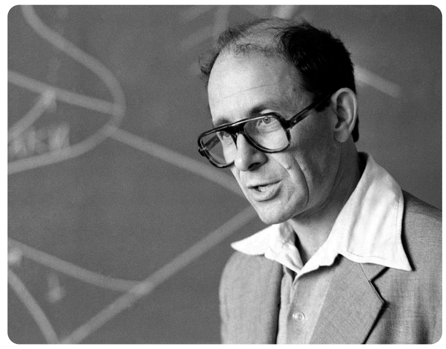
$\Rightarrow$ $S(\gamma) = \int_{t_0}^{t_1} \frac{1}{2} \langle \dot\gamma, \dot\gamma \rangle_\gamma \, dt$

$$\boxed{\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\gamma_\varepsilon) = 0 \Longrightarrow \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \ell(\gamma_\varepsilon) = 0}$$

Energy $E(\gamma(t), \dot\gamma(t)) = \frac{1}{2} \langle \dot\gamma(t), \dot\gamma(t) \rangle_{\gamma(t)}$ conserved

$\Rightarrow$ **constant speed parametrization**

Groups of diffeomorphisms



M compact Riemannian manifold
(think fluid domain)

Diffeomorphism:

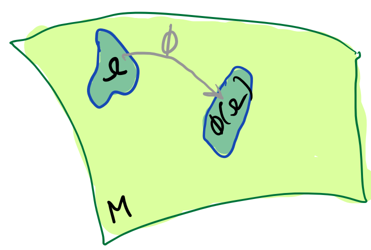
smooth bijection $\Phi : M \to M$ with smooth inverse Φ^{-1}

Diff(M) infinite dimensional Lie group:

- multiplication $(\Phi, \Psi) \mapsto \Phi \circ \Psi$
- identity $\mathrm{id} : x \mapsto x$
- inverse map $\Phi \mapsto \Phi^{-1}$

Volume preserving diffeomorphisms:

$$\mathrm{SDiff}(M) = \{ \Phi \in \mathrm{Diff}(M) \mid \Phi^* dx = dx \}$$

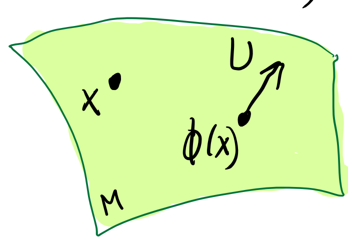


Arnold's idea:

Right-invariant Riemannian metric on SDiff(M)

$$\langle \dot\Phi, \dot\Phi \rangle_\Phi = \int_M \langle \dot\Phi(x), \dot\Phi(x) \rangle_{\Phi(x)} \, dx \qquad \dot\Phi : x \mapsto U \in T_{\phi(x)} M$$

$$= \int_M \langle \underbrace{\dot\Phi \circ \Phi^{-1}}_{V}, \underbrace{\dot\Phi \circ \Phi^{-1}}_{V} \rangle_x \, dx = \langle V, V \rangle_{L^2}$$



V divergence free vector field

$$\mathfrak{X}_{dx}(M) = \{ v \in \mathfrak{X}(M) \mid \operatorname{div} v = 0 \}$$

Lie algebra of SDiff(M)

SDiff(M)-symmetric Lagrangian

$$L(\Phi,\dot\Phi) = \frac{1}{2}\langle \dot\Phi\circ\Phi^{-1}, \dot\Phi\circ\Phi^{-1}\rangle_{L^2}$$

Corresponding Lie-Poisson system

Identify $\mathfrak{X}_{dx}(M)^* \simeq \mathfrak{X}_{dx}(M)$ (smooth dual)

Reduced Hamiltonian: $\bar H(v) = \frac{1}{2}\langle v,v\rangle_{L^2}$

kinetic energy of fluid

Recall Lie-Poisson:

$$\boxed{\dot v = ad^*_{d\bar H(v)} v}$$

$$d\bar H(v) = \frac{\delta \bar H}{\delta v} = v$$

$$\mathfrak{X}(M) = \mathfrak{X}_{dx}(M) \oplus \nabla C^\infty(M)$$

$$u = v + \nabla f$$

(Helmholz decomposition)

$$\Pi u = v$$

$$ad^*_v v = -\Pi(\nabla_v v) = -\nabla_v v - \nabla p$$

Lagrange multiplier

$$-\Delta p = \operatorname{div}(\nabla_v v)$$

$$\boxed{\dot v + \nabla_v v = -\nabla p}$$

Lagrangian coordinates (Euler-Lagrange on TSDiff(M))

$$\ddot\Phi = -\nabla p\circ\Phi$$

"fluid particles move as best they can along straight lines"