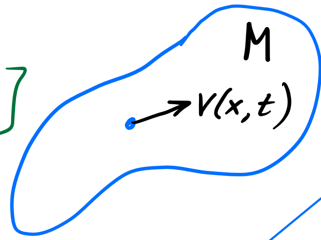
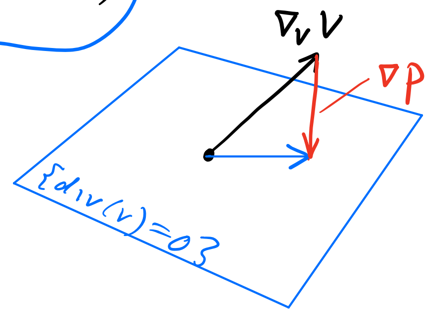
Perfect 2-D fluid [Euler 1757]





velocity

$$\dot v + \nabla_v v = -\nabla p, \quad div\, v = 0$$

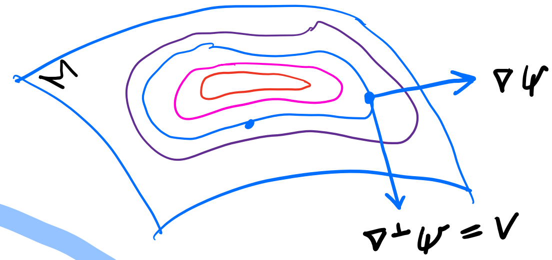
$$\Downarrow$$

streamfunction

vorticity

$$\dot\omega + \{\psi, \omega\} = 0$$

$$\Delta \psi = \omega$$



APPROACHES FOR LONG-TIME BEHAVIOR

statistical hydrodynamics [Onsager, 1949]
- disregards (most) dynamics
- discretization
- Liouville
- assume ergodic
- conservation laws

numerical simulations
- approximates dynamics (discretization)

PDE analysis
- exact dynamics
- rigorous results
- HARD!

matrix hydrodynamics (Zeitlin model)
- low local accuracy
- Liouville (symplectic)
- conservation laws

traditional numerics
- high local accuracy
- no Liouville
- no (or few) conservation laws

LONG-TIME QUALITATIVE BEHAVIOR

SHORT-TIME TRAJECTORY TRACKING

Hamiltonian description: Lie-Poisson dynamics

G : Lie group

Hamiltonian $\bar{H} : T^*G \to \mathbb{R}$ is right-invariant:

$$\bar{H}((q,p)\cdot g) = \bar{H}(q,p)$$

Symmetry group G : dynamics reduce to $T^*G/G \simeq \mathfrak{g}^*$

dual of Lie algebra

$$[(q,p)] \longleftrightarrow (q,p)\cdot q^{-1} = (e, \underbrace{p\cdot q^{-1}}_{m}) \simeq m \in T_e^*G = \mathfrak{g}^*$$

$$\bar{H}(q,p) = H(\underbrace{p\cdot q^{-1}}_{m}), \quad H : \mathfrak{g}^* \to \mathbb{R}$$

Hamilton's eq:

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial q} \end{cases} \xRightarrow{m \equiv p\cdot q^{-1}}$$

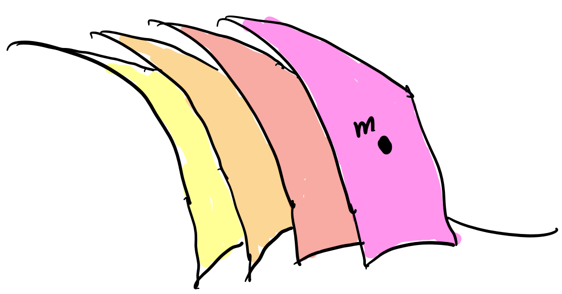
Lie-Poisson system

$$\boxed{\dot{m} - ad^*_{dH(m)} m = 0}$$

Preserves Poisson structure on $\mathfrak{g}^*$

$$\{dF, dG\}(m) = \langle m, [dF(m), dG(m)] \rangle$$

$\mathfrak{g}^*$ foliated in co-adjoint orbits (symplectic leafs)



$$O_m = \{ Ad_g^*(m) \mid g \in G \}$$

2-D Euler case

Lie group: $G = \mathrm{Diff}_\mu(M)$ ← area-preserving diffeomorphisms (= symplectomorphisms)

Lie algebra: $\mathfrak{g} = T_e \mathrm{Diff}_\mu(M) =$

$$= \{ v \mid \mathcal{L}_v \mu = 0 \} = \{ v \mid \operatorname{div} v = 0 \}$$

$$\overset{(*)}{=} \{ X_\psi \mid \psi \in C^\infty(M) \} \simeq C^\infty(M)/\mathbb{R}$$

streamfunction is Hamiltonian

Dual of Lie algebra: $\mathfrak{g}^* = \left(C^\infty(M)/\mathbb{R} \right)^*$

smooth part of dual: $C^\infty_0(M) \subset \mathfrak{g}^*$

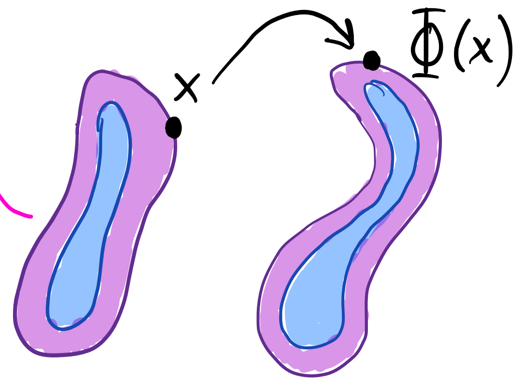
$$\left(\bar{\omega} : \mathfrak{g} \longrightarrow \mathbb{R}, \ \bar{\omega}(\psi) = \int_M \psi\, \omega\, \mu, \quad \omega \in C^\infty_0(M) \right)$$

Co-adjoint action:

$$\mathrm{Ad}^*_\Phi(\omega) = \omega \circ \Phi^{-1}$$

$$\Phi \in \mathrm{Diff}_\mu(M)$$

levelsets of ω



NOTE: $\bar{\omega} \simeq \omega \Rightarrow \mathrm{Ad}^*_\Phi(\bar{\omega}) \simeq \omega \circ \Phi^{-1}$
(smooth remain smooth)

Lie-Poisson eq: (for restriction to smooth dual)

$$H(\omega) = -\frac{1}{2} \int_M \omega \Delta^{-1} \omega \, \mu = -\frac{1}{2} \int_M \omega \psi \, \mu$$

$$\dot{\omega} - \left\{ \frac{\delta H}{\delta \omega}, \omega \right\} = 0 \iff \dot{\omega} + \{ \psi, \omega \} = 0, \quad \Delta \psi = \omega$$

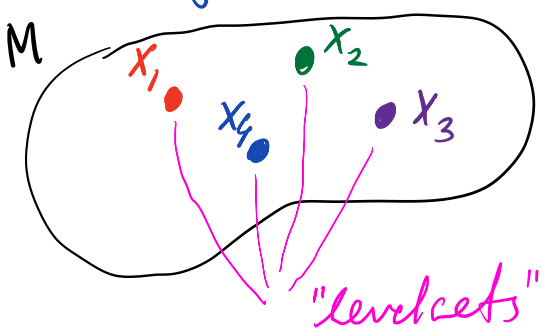
Casimir functions: $f : \mathbb{R} \to \mathbb{R}$

$$C_f(\omega) = \int_M f(\omega)\,\mu$$

PROOF: $C_f(Ad_\Phi^* \omega) = \int_M f(\omega \circ \Phi^{-1})\,\mu$

$$= \int_M f(\omega)\, \Phi^* \mu = C_f(\omega)$$

What if ω has singular support?



Formally: leave smooth dual

$$\bar{\omega}(\psi) = \Gamma_1 \psi(x_1) + \Gamma_2 \psi(x_2) + \Gamma_3 \psi(x_3) + \Gamma_4 \psi(x_4)$$

$$= \int_M \underbrace{\left(\sum_{k=1}^{N} \Gamma_k \delta_{x_k} \right)}_{\bar{\omega}} \psi$$

$$Ad_\Phi^*(\bar{\omega}) = \sum_{k=1}^{N} \Gamma_k \delta_{\Phi(x_k)}$$

Thus: singular co-adjoint orbit

$$O_{\bar{\omega}} = \left\{ \sum_{k=1}^{N} \Gamma_k \delta_{y_k} \;\middle|\; (y_1, \dots, y_N) \in M^N \right\} \simeq M^N$$

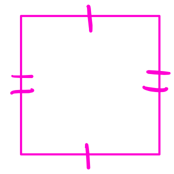
$\boxed{\text{Hamiltonian}}$: $H(x_1, \dots, x_N) = -\frac{1}{2} \int_M \bar{\omega}\, \Delta^{-1} \bar{\omega}$

$$= -\frac{1}{2} \int_M \sum_{k=1}^{N} \Gamma_k \delta_{x_k} \left(\sum_{\ell=1}^{N} \Gamma_\ell G(x_\ell, \cdot) \right) = \underline{-\frac{1}{2} \sum_{k,\ell} \Gamma_k \Gamma_\ell G(x_\ell, x_k)}$$

POINT VORTICES!

Onsager's brilliant idea: apply statistical mechanics to N PV system on $(\mathbb{T}^2)^N$

flat 2-torus



> When we compare our idealised model with reality, we have to admit one profound difference: the distributions of vorticity which occur in the actual flow of normal liquids (¹) are continuous, and in two-dimensional convection the vorticity of every volume element of the liquid is conserved, so that convective processes can build vortices only in the sense of bringing together volume elements of great initial vorticity.

Zeitlin's idea (1991)

Use quantization theory to approximate inf.-dim Lie-Poisson system by finite-dim Lie-Poisson

Quantization in a nutshell (for $M = S^2$)

Vorticity state $\omega \in C^\infty(S^2)$

skew Hermitian matrices

Aim: mapping $T_N : \omega \longmapsto W \in \mathfrak{u}(N)$ such that

$$T_N(\{\omega_1, \omega_2\}) \approx \frac{1}{\hbar}[T_N \omega_1, T_N \omega_2]$$

matrix commutator

$$\hbar_N = \frac{1}{\sqrt{N^2 - 1}}$$

Hoppe (1989): explicit quantization on $\mathbb{T}^2$ and $\mathbb{S}^2$

Route: (for $\mathbb{S}^2$) $\omega \longmapsto \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \omega_{\ell m} Y_{\ell m} \longmapsto \sum_{\ell=0}^{N}\sum_{m=-\ell}^{\ell} \omega_{\ell m} T^N_{\ell m}$, $T^N_{\ell m} \in \mathfrak{u}(N)$

spherical harmonics

Euler–Zeitlin equation

$$\omega \in C^\infty(\mathbb{S}^2) \longleftrightarrow W \in \mathfrak{u}(N)$$

$$\Delta\psi = \omega \longleftrightarrow \Delta_N P = W$$

Hoppe–Yau Laplacian (1998)

$$\dot\omega = \{\psi, \omega\} \longleftrightarrow \dot W = \frac{1}{\hbar}[P, W]$$

stream matrix   vorticity matrix

Dictionary Hydrodynamics — Matrix theory

"Classical" hydrodynamics	matrix hydrodynamics
vorticity $\omega \in C^\infty(\mathbb{S}^2)$	$W \in \mathfrak{u}(N)$
Casimir $C_f(\omega) = \int_{\mathbb{S}^2} f(\omega(x))\, \mu(x)$	$C^N_f(W) = \mathrm{tr}(f(W))$
Hamiltonian $H(\omega) = \frac{1}{2}\int_{\mathbb{S}^2} \psi\,\omega\, d\mu$	$H^N(W) = \frac{1}{2}\mathrm{tr}(PW)$
values of ω	eigenvalues of W
levelsets of ω	eigenvectors of W
$\lVert\omega\rVert_{L^\infty}$	spectral norm
average ω along levelsets of ψ	projection of W onto $\mathrm{stab}_P = \{W_s \in \mathfrak{u}(N) \mid [P, W_s] = 0\}$