EMMY NOETHER'S THEOREM

Invariante Variationsprobleme.

(F. Klein zum fünfzigjährigen Doktorjubiläum.)

Von

Emmy Noether in Göttingen.

Vorgelegt von F. Klein in der Sitzung vom 26. Juli 1918 [1]).

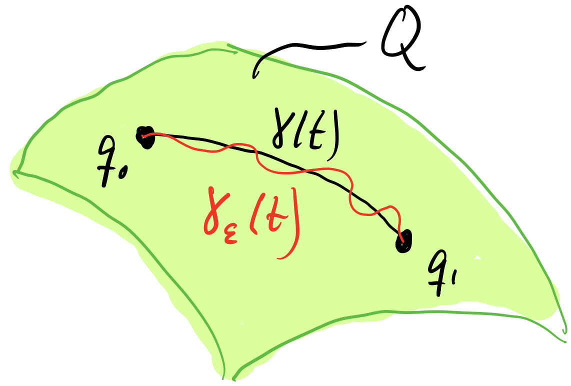
Es handelt sich um Variationsprobleme, die eine kontinuierliche Gruppe (im Lieschen Sinne) gestatten; die daraus sich er-

Felix Klein
David Hilbert
Pavel Alexandrov
(der Noether)

Variational mechanics (Lagrangian mechanics)

Setting:

Q: configuration space (manifold)



$L(q, \dot q, t)$: Lagrangian function ($TQ \times \mathbb{R} \longrightarrow \mathbb{R}$)

Hamilton's principle:

action $\quad S(\gamma) = \int_{t_0}^{t_1} L(\gamma(t), \dot\gamma(t), t)\, dt$ extremized

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S(\gamma_\varepsilon(t)) = 0 \quad \text{for } \gamma_0(t) = \gamma(t)$$

Euler-Lagrange equations:

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S(\gamma_\varepsilon(t)) = \int_{t_0}^{t_1}\left[\frac{\partial L}{\partial q}\cdot\underbrace{\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\gamma_\varepsilon(t)}_{\delta q} + \frac{\partial L}{\partial \dot q}\cdot\underbrace{\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\dot\gamma_\varepsilon(t)}_{\frac{d}{dt}\delta q}\right]dt =$$

$$= \int_{t_0}^{t_1}\left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\frac{d}{dt}\delta q\right]dt = \int_{t_0}^{t_1}\left[\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q}\right]\delta q\, dt + \overset{\nearrow 0}{\left[\frac{\partial L}{\partial \dot q}\delta q\right]_{t_0}^{t_1}}$$

$\Rightarrow$ **Euler-Lagrange eq:** $\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = 0}$ $(*)$

Special case:

$Q = \mathbb{R}^n$

$L(q,\dot q) = \frac{m}{2}|\dot q|^2 - V(q) \Rightarrow \boxed{m\ddot q = -\nabla V(q)}$ $(**)$

Newton's eq. for potential $V(q)$

Note: $(*)$ valid in <u>any</u> coordinates on Q

$(**)$ requires <u>affine</u> coordinates

<u>THEOREM</u> (Noether 1918)

Let $\Phi^s \in \mathrm{Diff}(Q\times\mathbb{R})$ be one-parameter subgroup s.t. $\Phi^s(q,t) = (\varphi^s(q,t), \eta^s(q,t))$

$$L(\varphi^s(q,t), D_q\varphi^s(q,t)\dot q, \eta^s(q,t)) = L(q,\dot q,t)$$

Then $I(q,\dot q) = \frac{\partial L}{\partial \dot q}\frac{\partial \varphi^s}{\partial s}\Big|_{s=0} - \left(\frac{\partial L}{\partial \dot q}\dot q - L\right)\frac{\partial \psi^s}{\partial s}\Big|_{s=0}$

is conserved along (*)

PROOF ($\psi^s(q,t) = t$)

$\varphi^s(q) \mapsto q_s$ $\qquad\qquad \varphi^s \in \mathrm{Diff}(Q)$

$\varphi^{s*}L = L$

$$L(q,\dot q,t) = L\Big(\underbrace{\varphi^s(q,t)}_{q_s}, \underbrace{D\varphi^s(q,t)\dot q}_{\dot q_s}, \underset{t_s}{t}\Big)$$

$$0 = \frac{d}{ds}L(q_s,\dot q_s,t) = \frac{\partial L}{\partial q}\cdot q_s' + \frac{\partial L}{\partial \dot q}\cdot \dot q_s'$$

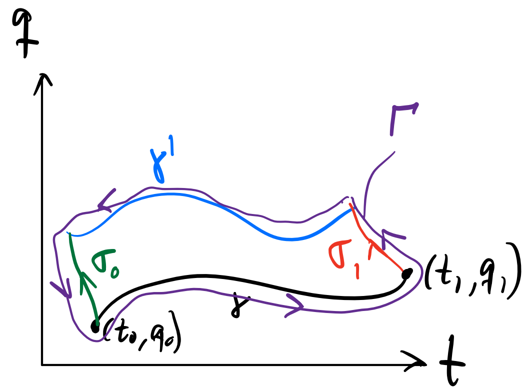
$\frac{\partial q_s}{\partial s}$

$$\boxed{\text{For any } s,\ q_s(t) \text{ solves EL: } \quad \frac{d}{dt}\frac{\partial L}{\partial \dot q_s} = \frac{\partial L}{\partial q_s}}$$

$$\Rightarrow \quad 0 = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot q_s}\right) q_s' + \frac{\partial L}{\partial \dot q}\left(\frac{d}{dt} q_s'\right)$$

$$= \frac{d}{dt}\Big(\underbrace{\frac{\partial L}{\partial \dot q_s} q_s'}_{I(q,\dot q)}\Big)$$

GEOMETRIC PROOF



$$S(q,t) = \int_{t_0}^{t} L(\gamma(\tau), \dot\gamma(\tau), \tau)\, d\tau$$

$$0 = \int_\Gamma dS = \underbrace{\int_\gamma dS - \int_{\gamma'} dS}_{0} + \int_{\sigma_1} dS - \int_{\sigma_0} dS$$

$$dS = \frac{\partial S}{\partial q}\cdot dq + \frac{\partial S}{\partial t}\, dt$$

$$\frac{d}{d\varepsilon} S(q+\varepsilon\delta q, t) = \int_{t_0}^{t} \cancel{\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q}\right)} \delta q + \left[\frac{\partial L}{\partial \dot q}\delta q\right]_{\cancel{t_0}}^{t}$$

$$= \frac{\partial L}{\partial \dot q}\delta q$$

$$\left.\begin{aligned} \frac{d}{dt} S &= L(q,t) \\ \frac{d}{dt} S &= \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q}\dot q \end{aligned}\right\} \Rightarrow \frac{\partial S}{\partial t} = \underbrace{L - \frac{\partial L}{\partial \dot q}\dot q}_{-E(q,\dot q)\ \text{energy}}$$

Along σ_1: $dt = \frac{\partial \psi}{\partial s} ds \qquad dq = \frac{\partial \varphi}{\partial s} ds$

σ_0: $dt = \frac{\partial \psi}{\partial s} ds \qquad dq = \frac{\partial \varphi}{\partial s} ds$

$$\Longrightarrow \quad \int_{\sigma_0} dS = \int_{\sigma_1} dS$$

$$\int_{\sigma_0} \frac{\partial S}{\partial q} dq + \frac{\partial S}{\partial t} dt = \int_{\sigma_1} \frac{\partial S}{\partial q} dq + \frac{\partial S}{\partial t} dt$$

$$\int_0^{s_1} \left(\frac{\partial L}{\partial \dot q} \frac{\partial \varphi}{\partial s} + \left(L - \frac{\partial L}{\partial \dot q} \dot q \right) \frac{\partial \eta}{\partial s} \right) ds = \int_0^{s_1} \left(\frac{\partial L}{\partial \dot q} \frac{\partial \varphi}{\partial s} + \left(L - \frac{\partial L}{\partial \dot q} \dot q \right) \frac{\partial \eta}{\partial s} \right) ds$$

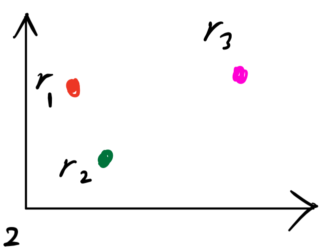
Example: $L(q, \dot q, t) = L(q, \dot q)$ (autonomous)

$$\left.\begin{aligned} \eta^s(q,t) &= -s \\ \frac{\partial \varphi}{\partial s} &= 0 \end{aligned}\right\} \Rightarrow E(q, \dot q) = -\left(\frac{\partial L}{\partial \dot q} \dot q - L \right) \frac{\partial \eta^s}{\partial s}$$

$$= \frac{\partial L}{\partial \dot q} \dot q - L$$ ← energy is conserved!

Example $Q = \mathbb{R}^{3N}$

$q = (r_1, \dots, r_N)$



$$L(q, \dot q) = \sum_{i=1}^{N} \frac{m_i |\dot r_i|^2}{2} - V(|r_1|, \dots, |r_N|)$$

$$L(e^{s\tilde\Omega} q, e^{s\tilde\Omega} \dot q) = L(q, \dot q) \qquad \tilde\Omega \in so(3)$$

$$I(q, \dot q) = \frac{\partial L}{\partial \dot q} \left.\frac{d}{ds}\right|_{s=0} e^{s\tilde\Omega} q = \sum_{i=1}^{N} m_i \dot r_i \cdot \tilde\Omega q$$

$$= \sum_{i=1}^{N} m_i r_i \cdot (\Omega \times \dot{r}_i) = \underbrace{\sum_{i=1}^{N} m_i (r_i \times \dot{r}_i)}_{\text{angular momentum}} \cdot \Omega$$

$\Omega \in \mathbb{R}^3$