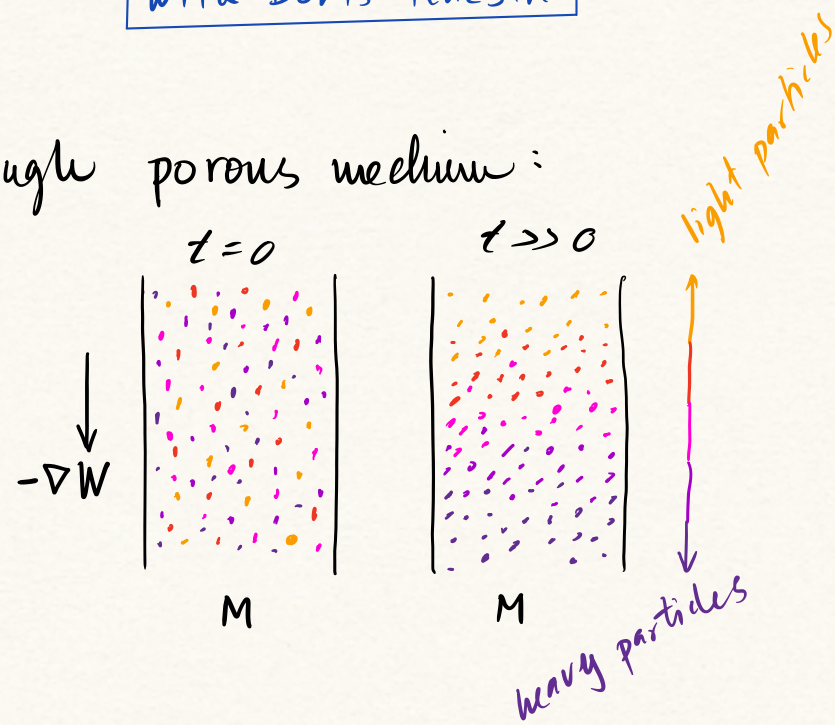
Gradient flows on Lie groups: Toda flow as incompressible porous medium

with Boris Khesin

IPM equation

incompressible flow through porous medium:

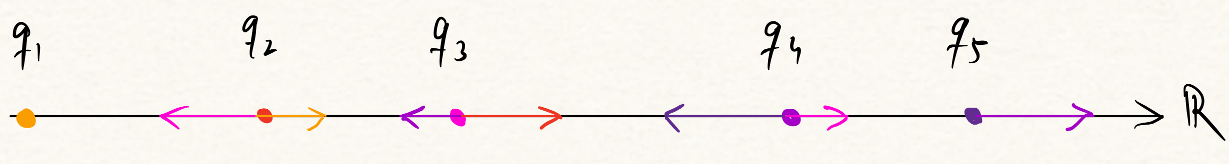
$$\begin{cases} \dot\rho + \operatorname{div}(\rho v) = 0 \\ v + \rho \nabla W = -\nabla p \\ \operatorname{div} v = 0 \end{cases}$$



Toda lattice

n particles on $\mathbb{R}$, neighbour interaction

$$H(q_1,\dots,q_n,p_1,\dots,p_n) = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{j=1}^{n-1} e^{2(q_j - q_{j+1})}$$



Claim: IPM & Toda describe gradient flows on group $\mathrm{Diff}_\mu(M)$ w.r.t

2-D

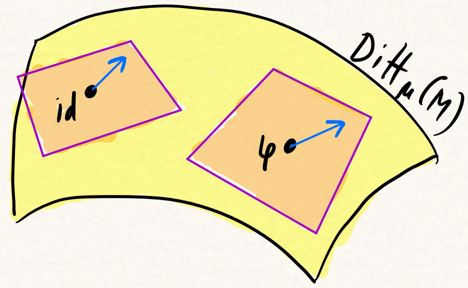
- same potential
- different Riemannian metrics (right-invariant)

Gradient flows on $\mathrm{Diff}_\mu(M)$

area form — 2-manifold (Riemannian symplectic)

Formally: $\dot\varphi = -\nabla_g E(\varphi) = -(\mathcal{G}_\varphi^\#)^{-1} dE(\varphi)$

Riemannian metric — energy functional



$$E(\varphi) = \int_M W(x)\, \omega_0 \circ \varphi^{-1}(x)\, \mu(x)$$

$$= F(\omega_0 \circ \varphi^{-1}) = F(\mathrm{Ad}^*_\varphi \omega_0)$$

$$\mathcal{G}_\varphi(\dot\varphi, \dot\varphi) = \int_M \langle \hat{A} v, v \rangle \mu \quad , \quad \hat{A} : \mathfrak{X}_\mu(M) \to \mathfrak{X}_\mu(M)^*$$

inertia operator — $v = \dot\varphi \circ \varphi^{-1}$ — what is this?

$$\begin{tikzcd}
\mathfrak{X}_\mu(M) \arrow[r, "\hat{A}"] & \mathfrak{X}_\mu(M)^* \simeq \Omega^1(M)/d\Omega^0(M) \arrow[d, "{[\alpha] \mapsto \underbrace{*d\alpha}_{\omega}}"] \\
C^\infty(M)/\mathbb{R} \arrow[u, "\psi \mapsto X_\psi"] \arrow[r, "A"] & C^\infty_0(M)
\end{tikzcd}$$

"curl $\hat{A}v$"

Lemma Gradient flow on $\mathrm{Diff}_\mu(M)$ w.r.t. A:

$$\dot\varphi = -X_\psi \circ \varphi \quad , \quad A\psi = \{W, \omega_0 \circ \varphi^{-1}\}$$

Or in terms of $\omega = \omega_0 \circ \varphi^{-1}$:

$$\dot\omega = -\{\underbrace{A^{-1}\{W, \omega\}}_{\psi}, \omega\}$$

double bracket

> Remark: general Lie group G
>
> $\dot m = \mathrm{ad}^*_{A^{-1}\mathrm{ad}^*_{dF(m)}(m)}(m)$

Back to IPM (2-D)

$$F(\rho)=\int_M W\rho\mu$$

$$\begin{cases}\dot\rho+\operatorname{div}(\rho v)=0\\ v+\rho\nabla W=-\nabla p\\ \operatorname{div} v=0\end{cases} \xRightarrow{v=X_\psi} \begin{cases}\dot\rho+\{\psi,\rho\}=0\\ \underbrace{-\Delta}_{A}\psi=\{W,\underbrace{\rho}_{\omega}\}\end{cases}$$

vorticity formulation

Remark: $(*)$ smooth ODE on $\mathrm{Diff}^s_\mu(M)$

proof:

$$\dot\varphi=((a\circ b\circ c\circ\tau)(\varphi^{-1}))\circ\varphi$$

$$\tau:\mathrm{Diff}^s_\mu(M)\to H^s(M)$$
$$\eta\longmapsto\rho_0\circ\eta$$

$$c: H^s(M)\longrightarrow H^{s-1}_0(M)$$
$$\rho\longmapsto\{W,\rho\}$$

$$b: H^{s-1}_0(M)\to H^{s+1}_0(M)$$
$$\rho\longmapsto\Delta^{-1}\rho$$

$$a: H_0^{s+1}(M)\longrightarrow \mathfrak{X}^s_\mu(M)$$
$$\psi\longmapsto\nabla^\perp\psi$$

$\Rightarrow$ ODE!

Smoothness: $\varphi\mapsto\varphi^{-1}$ not smooth but $P(\varphi^{-1})\circ\varphi$ is!

non-lin. diff operator

$$\begin{cases}\dot\varphi=X_\psi\circ\varphi,\\ \Delta\psi=-\{W,\rho_0\circ\varphi^{-1}\}\end{cases}\quad(*)$$

gradient formulation on $\mathrm{Diff}_\mu(M)$

Remark $(*)$ natural on any Kähler manifold $(M,J,\mu)\Rightarrow$ flow on <u>symplectomorphisms</u>

Remark Casimirs preserved

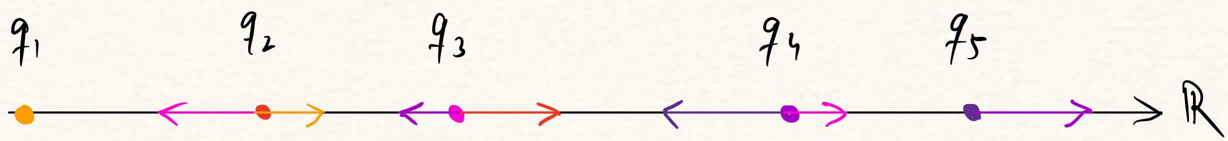
$$C_f(\rho)=\int_M f\circ\rho\,\mu,\ f:\mathbb{R}\to\mathbb{R}$$

ρ evolves on co-adjoint orbit

Back to Toda flow

Recall:

$$H(q_1,\dots,q_n,p_1,\dots,p_n)=\frac{1}{2}\sum_{j=1}^{n} p_j^2+\sum_{j=1}^{n-1} e^{2(q_j-q_{j+1})}$$



Flaschka, Moser, Bloch: new variables

$$a_j = e^{q_j-q_{j+1}}, \quad b_k = p_k$$

$$L=\begin{bmatrix} b_1 & a_1 & & \\ a_1 & b_2 & a_2 & \\ & a_2 & \ddots & \ddots \\ & & \ddots & b_n \end{bmatrix}$$

$$D=\operatorname{diag}(1,\dots,n)$$

$$\boxed{\dot{L}=[L,[L,D]]} \quad (*)$$

double bracket flow

framework on $SO(n)$:

gradient flow on orbits of Sym_n

$$R\cdot L = RLR^T \quad (\text{isospectral!})$$

Energy:

$$E(R)=F(RLR^T), \quad F(L)=tr(LD)$$

minimized on orbit when L diagonal with sorted eigenvalues

Continous Toda (dispersionless Toda)

$z \in \mathbb{R}$

Classical limit: $\{a_j^n\}_{j=1}^{n-1}, \{b_k^n\}_{k=1}^{n} \longrightarrow a(z), b(z)$

$L \in Sym_n \longrightarrow \ell(\phi, z)$ $\quad (\phi, z) \in T^*S^1$

cylinder

as $\hbar = \frac{1}{n} \to 0$

$$\ell(\phi, z) = b(z) + 2a(z)\cos\phi$$

$$\boxed{\dot{\ell} = \{\ell, \{\ell, z\}\}} \qquad (*)$$

Remark
$(*)$ <u>not</u> ODE on Diff^s_μ

$(*)$ is "vorticity" formulation for gradient flow on $\mathrm{Diff}_\mu(T^*S^1)$

$$F(\ell) = \frac{1}{2}\int_{T^*S^1} \ell\, z \, d\phi \wedge dz = \langle \ell, z \rangle_{L^2(T^*S^1)}$$

COMPARISON

	TODA	IPM
classical	$\dot{\ell} = \{\underbrace{\ell}_{A^{-1}}, \{\ell, \underbrace{z}_{W(\phi,z)}\}\}$	$\dot{\rho} = \{\rho, \underbrace{-\Delta^{-1}}_{A^{-1}}\{\rho, \underbrace{z}_{W(\phi,z)}\}\}$
quantized (discretized)	$L = [L, [L, D]]$ $L \in Sym_n$	$\dot{P} = [P, -\underbrace{\Delta_n}_{\text{Hoppe–Yau quantized Laplacian}}^{-1}[P, D]]$

NUMERICAL EXPERIMENTS

Initial configuration $\ell_0 \in C^\infty(S^2)$