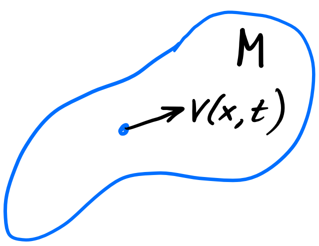
Matrix hydrodynamics and 2-D Euler

Klas Modin, EquaDiff 2024, Karlstad

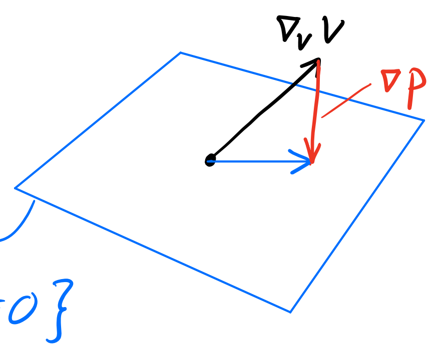
Dedicated to Vladimir Zeitlin

The Father of Matrix Hydrodynamics

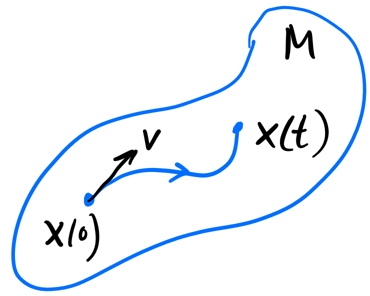
Incompressible inviscid hydrodynamics (Euler 1757)



$$\begin{cases} \dot{v} + \nabla_v v = -\nabla p \\ \operatorname{div} v = 0 \end{cases}$$



Arnold's viewpoint



Integrate: $\dot{x}(t) = v(x(t),t)$

Flow map (all $x \in M$ at once):

$$\dot\varphi_t = v_t \circ \varphi_t \,, \qquad \varphi_0(x) = x$$

tracks fluid particles

NOTICE: $\varphi_t \in \mathrm{Diff}_\mu(M) = \{\eta \in \mathrm{Diff}(M) \mid \eta_* \mu = \mu\}$

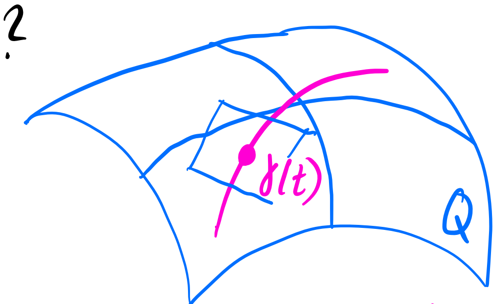
Newton's equation (differentiate)

$$\ddot\varphi = \dot v \circ \varphi + \nabla v \circ \varphi \cdot \underbrace{\dot\varphi}_{v\circ\varphi} = (\dot v + \nabla_v v)\circ\varphi$$

$$= (-\cancel{\nabla_v v} - \nabla p + \cancel{\nabla_v v})\circ\varphi$$

$$\Rightarrow \quad \ddot\varphi = -\nabla p \circ \varphi \qquad (*)$$

What is a geodesic curve?



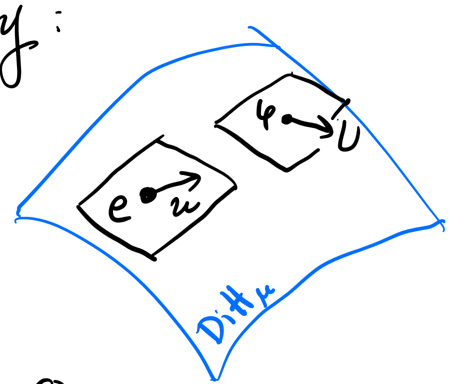
Idea: take $Q = \mathrm{Diff}_\mu(M)$

inf. dim manifold

$\ddot\gamma \in (T_\gamma Q)^\perp$

Is $\nabla p \circ \varphi \perp$ to $T_\varphi \mathrm{Diff}_\mu \subset T_\varphi \mathrm{Diff}$?

Riemannian metric: kinetic energy:



$$\langle U, U\rangle_\varphi = \int_M |U(x)|^2 \, d\mu(x)$$

$$U \in T_\varphi \mathrm{Diff}_\mu(M) \iff \mathrm{div}\big(\underbrace{U\circ\varphi^{-1}}_{u}\big) = 0$$

$$\begin{aligned} U : M &\longrightarrow TM \\ x &\longmapsto U(x) \in T_{\varphi(x)}M \end{aligned}$$

Back to Euler equations: $\ddot{\varphi} = -\nabla p \circ \varphi$

Thm (Arnold 1966)

$\varphi(t) \in \mathrm{Diff}_\mu(M)$ is geodesic curve

proof:

$$\langle U, \nabla p\circ\varphi\rangle_\varphi = \int_M U\cdot\nabla p\circ\varphi \, d\mu = \int_M (u\cdot\nabla p)\circ\varphi \, d\mu$$

$$= \int_M u\cdot\nabla p \, \underbrace{\varphi_* d\mu}_{d\mu} = \underbrace{\int_M u\cdot\nabla p \, d\mu}_{\langle u, \nabla p\rangle_e = \langle u, \nabla p\rangle_{L^2}} = 0 \qquad \square$$

NOTE: Metric is <u>right-invariant</u>: $\langle U\circ\eta, V\circ\eta\rangle_{\varphi\circ\eta} = \langle U, V\rangle_\varphi$

Hamiltonian description: Lie-Poisson dynamics

G : Lie group

Hamiltonian $\bar{H} : T^*G \to \mathbb{R}$ is right-invariant:

$$\bar{H}((q,p)\cdot g) = \bar{H}(q,p)$$

Symmetry group G : dynamics reduce to $T^*G/G \simeq \mathfrak{g}^*$

dual of Lie algebra

$$[(q,p)] \longleftrightarrow (q,p)\cdot q^{-1} = (e, \underbrace{p\cdot q^{-1}}_{m}) \simeq m \in T_e^*G = \mathfrak{g}^*$$

$$\bar{H}(q,p) = H(\underbrace{p\cdot q^{-1}}_{m}), \quad H : \mathfrak{g}^* \to \mathbb{R}$$

Hamilton's eq:

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial q} \end{cases} \xRightarrow{m \equiv p\cdot q^{-1}}$$

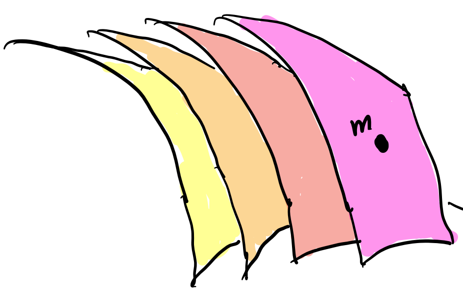
Lie-Poisson system

$$\boxed{\dot{m} - ad^*_{dH(m)} m = 0}$$

Preserves Poisson structure on $\mathfrak{g}^*$

$$\{dF, dG\}(m) = \langle m, [dF(m), dG(m)] \rangle$$

$\mathfrak{g}^*$ foliated in co-adjoint orbits (symplectic leafs)



$$O_m = \{ Ad^*_g(m) \mid g \in G \}$$

Special case M Riemannian (compact)

$$G = \mathrm{Diff}_\mu(M) \quad \text{(Frechet Lie group)}$$

$$\mathfrak{g} = \mathfrak{X}_\mu(M) = \{ v \in \mathfrak{X}(M) \mid \operatorname{div} v = 0 \}$$

$$\underbrace{\mathfrak{g}^*}_{\text{smooth dual}} \simeq \Omega^1(M)/d\Omega^0(M) \underset{\text{H trivial 1-co-homology}}{\simeq} d\Omega^1(M) \quad \text{(exact 2-forms)}$$

Hamiltonian on $\mathfrak{g}^* \simeq d\Omega^1(M)$

$$H(\underbrace{\hat\omega}_{\text{exact 2-form}}) = \frac{1}{2}\int_M i_v d^{-1}\hat\omega\,\mu, \quad dv^\flat = \hat\omega, \quad \operatorname{div} v = 0$$

2-D case: $\hat\omega = \omega \underbrace{\mu}_{\text{symplectic structure on } M}$

$$v = X_{\underbrace{\psi}_{\text{Hamiltonian on } M}} = \nabla^\perp \psi$$

$$H(\omega) = \frac{1}{2}\int_M \psi\,\omega\,\mu, \quad -\Delta\psi = \omega$$

$$\langle \mathrm{ad}^*_{X_\psi}\hat\omega, X_\xi \rangle = \langle \omega, \underbrace{\{\psi, \xi\}}_{\text{Poisson bracket on } M} \rangle_{L^2} = \langle -\{\psi, \omega\}, \xi \rangle_{L^2}$$

$$\dot{\hat\omega} - \mathrm{ad}^*_{X_\psi}\hat\omega = 0 \iff \boxed{\dot\omega + \{\psi, \omega\} = 0 \quad (*)}$$

$$\iff \dot\omega + \mathcal{L}_{X_\psi}\omega = 0 \iff \dot\omega + \operatorname{div}(\omega \nabla^\top \psi) = 0$$

Casimirs: $C_f(\omega) = \int_M f \circ \omega \, \mu$ preserved for each $f : \mathbb{R} \to \mathbb{R}$

Proof: C_f constant on co-adjoint orbits

Zeitlin's idea (1991)

Use **quantization theory** to replace (*) by finite dim. Lie-Poisson system

Quantization in a nutshell (for $M = S^2$)

Vorticity state $\omega : S^2 \to \mathbb{R}$

Aim: mapping $T_N : \omega \longmapsto W \in \mathfrak{u}(N)$ such that

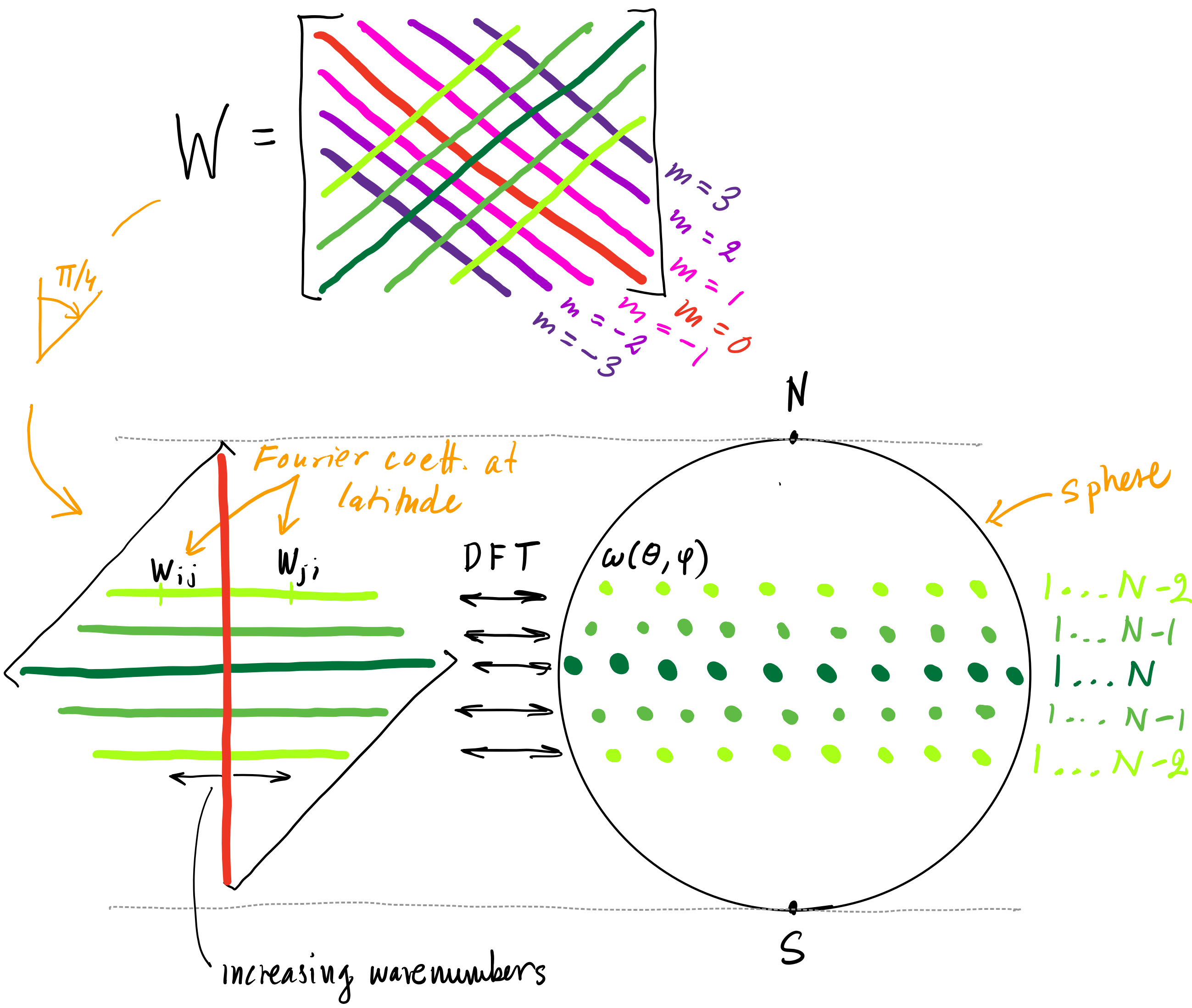
$$T_N(\{\omega_1, \omega_2\}) \approx \frac{1}{\hbar}[T_N \omega_1, T_N \omega_2]$$ ← matrix commutator

$$\hbar_N = \frac{1}{\sqrt{N^2 - 1}}$$

Hoppe (1989): explicit quantization on $\mathbb{T}^2$ and S^2

Route: (for S^2)
$$\omega \longmapsto \underbrace{\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \omega_{\ell m} Y_{\ell m}}_{\text{spherical harmonics}} \longmapsto \sum_{\ell=0}^{N} \sum_{m=-\ell}^{\ell} \omega_{\ell m} \underbrace{T^N_{\ell m}}_{\in\, \mathfrak{u}(N)}$$

What is the Matrix? (Neo, 1999)



Euler-Zeitlin equation

$$\omega \in C^\infty(S^2) \longleftrightarrow W \in \mathfrak{u}(N)$$

$$\Delta \psi = \omega \longleftrightarrow \Delta_N P = W$$

Hoppe-Yau Laplacian (1998)

$$\dot{\omega} = \{\psi, \omega\} \longleftrightarrow \dot{W} = \frac{1}{\hbar}[P, W]$$

stream matrix

vorticity matrix

Dictionary Hydrodynamics — Matrix theory

"Classical" hydrodynamics	matrix hydrodynamics
vorticity $\omega \in L^\infty(S^2)$	$W \in \mathfrak{u}(N)$
Casimir $I_f(\omega) = \int_{S^2} f(\omega(x))\, d\mu(x)$	$I_f^N(W) = \mathrm{tr}(f(W))$
Hamiltonian $H(\omega) = \frac{1}{2}\int_{S^2} \psi\, \omega\, d\mu$	$H^N(W) = \frac{1}{2}\mathrm{tr}(PW)$
values of ω	eigenvalues of W
levelsets of ω	eigenvectors of W
$\lVert \omega \rVert_{L^\infty}$	spectral norm
average ω along levelsets of ψ	projection of W onto $\mathrm{stab}_P = \{ W_s \in \mathfrak{u}(N) \mid [P, W_s] = 0 \}$